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PRIMARY DECOMPOSITION OF LEFT N-MODULES OVER RIGHT NEAR-RINGS

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ABSTRACT

In this paper we introduce the concepts of essential extension and rational extension to N-modules which are generalization of the notions essential extension and rational extension for modules over rings. Hans H. Storrer has introduced the notion of critical left ideals in the theory of rings, and has given a generalization of primary decomposition to non-commutative rings. In this paper we extend the notion of critical left ideals to near-rings. The relation between critical left ideals and left ideals of type 0 is studied in terms of the concepts essential extension and rational extension. Also we extend the notions of Primary, Co primary and Z-S Co primary for N-modules and some interesting results are obtained. All the near-rings are assumed to be zero-symmetric right near-rings with throughout this paper. Also the N-modules under consideration are all left-modules which are unital.

KEY WORDS

Left N-Modules over Right Near-Rings and Concepts essential extension and rational extension.

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INTRODUCTION

All the near-rings under consideration throughout this thesis are assumed to be zero-symmetric near-rings and also containing multiplicative identity. Further it is assumed that all the N-modules are unitary. The concepts of essential extension and rational extension were introduced by Barua (2) in the theory of near-rings and N-modules which already exist in the theory of rings and modules over rings. He introduced these concepts in terms of N-maps and N-sets. Hans. H. Storrer has introduced the notion of critical left ideals in the theory of rings, and has given a generalization of primary decomposition to non-commutative rings. In this chapter we extend the notion of critical left ideals to near-rings. The relation between critical left ideals of type 0 is studied in terms of the concepts essential extension and rational extension. Also we extend the notions of Primary, Co primary and Z-S co primary for N-modules and some interesting results are obtained[1-5].

Lawton proved that if N is a distributive generated near-ring having a faithful irreducible left N-module and if N has descending chain condition on left ideals then N contains all mappings of M into itself.

SECTION -1

Definition 1.1

Let N be a near-ring and let M be a left N-module and let Δ be a non-zero N-sub module of M. We say that M is a "module essential extension" of Δ if for every non-zero N-ideal Δ' of M, Δ∩ Δ' ≠0.

Definition 1.2

Let M be a left N-module over a right near-ring N. Let Δ be a non-zero N-sub module of M. We say that M is an essential extension of Δ if for every non-zero N-sub module Δ' we have Δ ∩ Δ' ≠ 0.

Remark 1.3

If M is an essential extension of Δ, then M is a module essential extension of Δ. But the converse need not be true.

Example 1.4

Let N = {0, a, b, c} be Klien four group under addition. The multiplication is defined by the following table.

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Take M = N and Δ = {0, a}. Now Δ is a non-zero N-sub module of M and M is module essential extension of Δ but not an essential extension of Δ since Δ□ = {0, b} is an N-sub module of M such that Δ∩Δ□ = {0}.

Definition 1.5

Let M be a left N-module and Δ be a non-zero N-sub module of M. We say that M is a rational extension of Δ if for any N-sub module Δ□ of M such that Δ ⊆ Δ□ ⊆ M and for any N-homomorphism f: Δ□→M such that Δ ⊂ Ker f, we have f = 0.

We know that if M is a module over a ring then the rational extension property of M implies essential extension property. But in the case of near-ring modules this need not be true which can be justified with the following example.

Example 1.6

Let D_8 = {0, a, 2a, 3a, b, a + b, a + 2b, a + 3b} be Dihedral group of order 8.

The addition and multiplication in D_8 are defined by the following tables.

.	**0**	**a**	**2a**	**3a**	**b**	**a+b**	**2a+b**	**3a+b**
0	0	a	2a	3a	b	a+b	2a+b	3a+b
a	a	2a	3a	0	a+b	2a+b	3a+b	b
2a	2a	3a	0	a	2a+b	3a+b	b	a+b
3a	3a	0	a	2a	3a+b	B	a+b	2a+b
b	b	3a+b	2a+b	a+b	0	3a	2a	a
a+b	a+b	b	3a+b	2a+b	a	0	3a	2a
2a+b	2a+b	a+b	b	3a+b	2a	A	0	3a
3a+b	3a+b	2a+b	a+b	b	3a	2a	a	0

.	**0**	**a**	**2a**	**3a**	**b**	**a+b**	**2a+b**	**3a+b**
0	0	0	0	0	0	0	0	0
a	0	a	2a	3a	b	a+b	2a+b	3a+b
2a	0	2a	0	2a	0	0	0	0
3a	0	3a	2a	a	b	a+b	2a+b	3a+b
b	0	b	2a	b	b	a+b	2a+b	3a+b
a+b	0	a+b	0	3a+b	0	0	0	0
2a+b	0	2a+b	2a	2a+b	b	a+b	2a+b	3a+b
3a+b	0	3a+b	0	a+b	0	0	0	0

Take N = M = D_8. Now N is a near-ring with identity a and M is left N-module. Take $\Delta = \{0, a + b\}$. Now we shall prove that M is a rational extension of Δ. Clearly Δ is an N-sub module of M. Let Δ' be any N-sub module of M such that $\Delta \subseteq \Delta' \subseteq M$. suppose f: $\Delta' \rightarrow M$ is an N-homomorphism such that $\Delta \subset \text{Ker } f$. Now here $\Delta \subset \text{Ker } f \subseteq \Delta'$. But the only proper N-sub module of M are {0,2a}, {0,b}, {0,a + b}, {0,2a + b}, {0,3a + b}, {0,b,2a,2a + b}. If Δ = Δ', then f = 0 since $\Delta \subset$ Ker f. If $\Delta \neq \Delta'$, then Δ' = M. Since $\Delta \subset$ Ker f and Ker f is an N-ideal of M, we have Ker f = Δ' = M. Hence f = 0. Thus M is a rational extension of Δ.

Suppose Δ' is a non-zero N-sub module of M which is a proper N-module of M different from Δ. Now $\Delta \cap \Delta' = \{0\}$. Therefore M is not essential extension of Δ. In fact M is not an essential extension of any of its proper N-sub modules.

Definition 1.7

A left N-module over a near-ring N is said to be uniform if it is an essential extension of each of its non-zero N-sub modules.

Definition 1.8

A left N-module M over a near-ring N is said to be strongly uniform if it is uniform and is a rational extension of each of its non-zero N-sub modules.

If X is a subset of an N-module M then < X > stands for the N-ideal of M generated by X i.e. the smallest N-ideal of M containing X.

Definition 1.9

We say that a left N-module M satisfies the intersection property if for any two N-sub modules Δ_1, Δ_2 of M we have

$< \Delta_1 \cap \Delta_2 > = < \Delta_1 > \cap < \Delta_2 >$.

Remark 1.10

If N is any near-ring such that every N-sub module of N is an N-ideal of N, then N-module satisfies the intersection property.

Proposition 1.11

Suppose M is a left N-module with ascending chain condition on N-sub modules and satisfying the intersection property. Then M has an N-ideal which is uniform.

Proof

Suppose if possible M has no N-ideal which is uniform. This implies M is not an essential extension of some non-zero N-sub module. Let Δ_1 be a non-zero N-sub module of M such that M is not an essential extension of Δ_1. Hence there exists a non-zero N-sub module Δ_1' such that $\Delta_1 \cap \Delta_1' = \{0\}$. By the intersection property we have

$< \Delta_1 > \cap < \Delta_1' > = < \Delta_1 \cap \Delta_1' > = \{0\}$.

Put $< \Delta_1 > = l_1$, and $< \Delta_1' > = l_1'$.

This implies $l_1 \cap l_1' = \{0\}$. By our assumption l_1 and l_1' are not uniform. Since l_1' is not uniform there exists a non-zero N-sub module Δ_2 of l_1' such that l_1' is not an essential extension of Δ_2. This implies there exists a non-zero N-sub module Δ_2' of l_1' such that $\Delta_2 \cap \Delta_2' = \{0\}$.

Put $< \Delta_2 > = l_2$ and $< \Delta_2' > = l_2'$. By intersection property we have

$< \Delta_2 > \cap < \Delta_2' > = < \Delta_2 \cap \Delta_2' > = \{0\} \Rightarrow l_2 \cap l_2' = \{0\}$.

Again by our assumption l_2 and l_2' are not uniform. Proceeding in this way we obtain a sequence $\{l_n\}$ of N-ideals of M such that each of them is not uniform and each l_n is an N-ideal contained in l'_{n-1} and $l_n \cap l'_n = \{0\}$ for every n. Since $l_1 \cap l_1' = \{0\}$ and $l_2 \subseteq l_1'$ we have $l_1 \cap l_2 = \{0\}$ which implies that l_2 is not contained in l_1. Hence $l_1 + l_2$ contain l_1 properly. Similarly we have $l_1 + l_2 + \ldots + l_n$ contains $l_1 + l_2 + \ldots + l_{n-1}$ properly. Thus we get an ascending sequence of N-sub modules $l_1 \subset l_1 + l_2 \subset l_1 + l_2 + l_3 \subset \ldots$ which is not stationary. This is a contradiction to the fact that M satisfies A.C.C. for N-sub modules. Therefore M has an N-ideal which is uniform.

2. Critical Left Ideals

Following the notion of critical left ideal as given by Storrer [14] in the theory of rings the concept of critical left ideals in near-rings is introduced.

Definition 2.1

A left ideal P of a near-ring N is called critical left ideal if M = N/P is strongly uniform N-module.

Remark 2.2

A left ideal of L of N is said to be modular if there exists e Є N such that n-ne Є L for every n Є N. If N is a near-ring with identity then every left ideal of N will be a modular left ideal.

Definition 2.3

A left ideal L of a near-ring N is of type 0 if M = N/L is an N-module which has no non-trivial N-ideals.

Definition 2.4

A left ideal L of a near-ring N is of type 2 if M = N/L is an N-module which has no non-trivial N-sub modules.

From the definition it follows that every left ideal of type 2 is a critical left ideal. However we can still strengthen the result as follows.

Proposition 2.5

Every left ideal of N which is of type 0 is a critical left ideal.

Proof: Let L be any left ideal of N of type 0.

To show that N/L is uniform, it is enough to show that the intersection of any two non-zero N-sub modules is non-zero. Suppose $\Delta_1 = L_i/l$, i = 1, 2 are two non-zero N-sub modules of M = N/L. Assume that $L_1 \not\subset L_2$ and $L_2 \not\subset L_1$. Choose $r_1 \in L_1$ such that $r_1 \notin L_2$ and $r_2 \in L_2$ such that $r_2 \notin L_1$. Since L is a modular left ideal of type 0, (L: N) = P is a 0-primitive ideal of N and M = N/L is an N/P-module of type 0.

By density theorem (Pilz, 115[12]) there exists $n_1 + P \in N_1/P$ such that $(n_1 + P)(r_2 + L) = r_1 + L$ which implies that $n_1r_2 - r_1 \in L \subset L_1$. Consequently $n_1r_2 \in L_1 \cap L_2$ and $n_1r_2 \notin L$. Thus $n_1r_2 + L \in \Delta_1 \cap \Delta_2$. This shows that intersection of any two non-zero N-sub modules of N/L is non-zero. That is M = N/L is uniform.

Now to establish that M = N/L is a rational extension of each of its non-zero N-sub modules, assume that there exists two non-zero N-sub modules Δ_1 and Δ_2 of M such that $\Delta_1 \subset \Delta_2$ and an N-homomorphism f: $\Delta_2 \rightarrow M$ with Ker f $\supset \Delta_1$. Suppose that $\Delta_1 = L_1/L$ and $\Delta_2 = L_2/L$. If $L_1 = L_2$, then clearly f = 0. Otherwise choose any $r_2 \in L_2$ such that $r_2 \notin L$ and $r_1 \in L_1$, such that $r_1 \notin L$. Considering M= N/L as an N/P-module and applying the density theorem, there exists $n_1 + P \in N_1/P$ such that $(N_1 + P)(r_1 + L) = r_2 + L$ which implies $n_1r_1 - r_2 \in L$. Since f is an N-homomorphism and $n_1r_1 + L \in \Delta_1$ we have $f(r_2 + L) = f(n_1r_1 + L) = 0$. That is $f(r_2 + L) = 0$ for every $r_2 \in L_2$. Therefore f = 0 and M is a rational extension of Δ_1. Hence L is a critical left ideal of N.

In the class of critical left ideals of a near-ring N, we introduce a relation as follows.

Definition 2.6

If l and J are two left ideals of N, l is said to be related to J if there exists $a \notin l$, $b \notin J$ such that $la^{-1} = Jb^{-1}$ where

$$la^{-1} = \{r \in N : ra \in l\} \text{ and}$$
$$Jb^{-1} = \{r \in N : rb \in J\}$$

Lemma 2.7

Let N be a near-ring and P and Q are two critical left ideals of N then the following statements are equivalent.

i) P is related to Q

ii) A non-zero N-sub module of N/P is isomorphic to a non-zero N-sub module of N/Q.

Proof

Suppose P and Q are related, then there exists a and b such that $Pa^{-1} = Qb^{-1}$ and $a \notin P$ and $b \notin Q$.

Consider $Pa^{-1} = \{r \in N : ra \in P\}$

It is a left ideal of N. If $Pa^{-1} = N$, then $1 \in Pa^{-1}$, this implies 1.a = a Є P which is not the case. Therefore $Pa^{-1} \neq N$. Also $Qb^{-1} \neq N$.

Define $\psi: N/Pa^{-1} \rightarrow N/P$

By $\psi(x + Pa^{-1}) = xa + P$ for x Є N.

We shall now show this mapping is well defined.

Suppose $x_1 + Pa^{-1} = x_2 + Pa^{-1}$ for some x_1, x_2 in N

⇨ $x_1 - x_2 \in Pa^{-1}$
⇨ $(x_1 - x_2)a \in P$
⇨ $x_1a - x_2a \in P$
⇨ $x_1a + P = x_2a + P$
⇨ $\Psi(x_1a + P) = \psi(x_2 a + P)$

Therefore ψ is well defined.

Suppose $\psi(x_1 + Pa^{-1}) = \psi(x_2 + Pa^{-1})$

⇨ $x_1a + P = x_2a + P$
⇨ $x_1a - x_2a \in P$
⇨ $(x_1 - x_2)a \in P$
⇨ $x_1 - x_2 \in Pa^{-1}$
⇨ $x_1 + Pa^{-1} = x_2 + Pa^{-1}$

Therefore ψ is one-one.

Hence ψ is an isomorphism of N/Pa^{-1} onto an N-submodule of N/P. Thus there is an isomorphism of N/Pa^{-1} onto an N-sub module L/P of N/P.

Similarly there is an isomorphism of N/Qb^{-1} onto an N-sub module L'/Q of N/Q. Since $Pa^{-1} = Qb^{-1}$ we have L/P is isomorphic to L'/Q where L/P and L'/Q are non-zero N-sub modules of N/P and N/Q respectively.

Conversely suppose that a non-zero N-sub module L_1/P of N/P is isomorphic to a non-zero N-sub module L_2/Q of N/Q. Let Φ be the isomorphism of L_1/P onto L_2/Q. Since L_2/Q is non-zero we have $\Phi \neq 0$. Let b + Q be a non-zero element of L_2/Q. This implies there exists a non-zero element a + P in L_1/P such that Φ (a + P) = b + Q. since a + P and b + Q are non-zero elements we have $a \notin P$ and $b \notin Q$.

Now we Show that $Pa^{-1} = Qb^{-1}$. Let $r \in Pa^{-1}$.

⇨ $ra \in P$
⇨ $ra + P = 0$
⇨ $\Phi(ra + P) = 0$
⇨ $\Phi(r(a + P)) = 0$
⇨ $r\Phi(a + P) = 0$
⇨ $r(b + Q) = 0$
⇨ $rb + Q = 0$
⇨ $rb \in Q$
⇨ $r \in Qb^{-1}$

Hence $Pa^{-1} \subseteq Qb^{-1}$. Similarly $Qb^{-1} \subseteq Pa^{-1}$. Therefore $Pa^{-1} = Qb^{-1}$ where $a \notin P$ and $b \notin Q$

Hence P is related to Q.

Proposition 2.8

The relation P is related to Q is an equivalence relation in the class of all critical left ideals.

Proof

The only condition to be verified is the following. If l, J, L are critical left ideals such that l is related to J and J is related to L then l is related to L.

By lemma 1.2.7, there exist non-zero N-sub modules Δ_1/l, Δ_2/J of N/l and N/J respectively such that

$\Delta_1/l \cong \Delta_2/J$

Similarly there exist non-zero N-sub modules Δ_3/J and Δ_4/L of N/J and N/L respectively such that $\Delta_3/J \cong \Delta_4/L$. Since N/J is uniform, we have $\Delta_2 \cap \Delta_3 \supset J$. Put $\Delta = \Delta_2 \cap \Delta_3$. Then Δ/J is isomorphic to a non-zero N-sub-module of N/l and also Δ/J is isomorphic to a non-zero N-sub module of N/L. Hence a non-zero N-sub module of N/L. Hence l is related to L.

The equivalence class containing P is denoted by [P].

3. Associated Left Ideals of a Module M

Definition 3.1

A critical left ideal P of N is said to belong to M if there exists $0 \neq x \in M$ such that Ann (x) = P.

Remark 3.2

The class of all critical left ideals related to P is denoted by [P].

Further the set all critical left ideals belonging to M is denoted by Ass M.

Theorem 3.2

If M satisfies ascending chain condition of N-sub modules and also satisfies intersection property. Then there exists a non-zero N-sub module B of M which is strongly uniform.

Proof: By R is strongly uniform there is nothing to prove. Suppose now R is not strongly uniform. But by our choice R is uniform. Therefore R cannot be a rational extension of each of its non-zero N-sub modules.

That is, there exists a maximal N-submodule A of R such that R is not a rational extension of A (since M satisfies A.C.C. on N-sub modules). Hence there exists N-sub module R' such that $A \subseteq R' \subseteq R$, a non-zero homomorphism Φ: R'→R such that Ker Φ contains A properly.

Let $\Phi(R') = B \neq 0$.

Then B is a non-zero N-sub module of R. We claim that B is a rational extension of each of its non-zero N-sub modules.

Let B' ≠ 0 be any N-sub module of B.

Put $\Phi^{-1}(B') = A'$. Then $A \subset A' \subseteq R$ and R is a rational extension of A' (by the maximality of A).

Let f: B″→B be a homomorphism where $B' \subseteq B'' \subseteq B$ with Ker f ⊃ B'

If $\Phi^{-1}(B'') = A''$ then $\Phi^{-1}(B) \supseteq \Phi^{-1}(B'') \supseteq \Phi^{-1}(B')$

=> $\Phi^{-1}(B) \supseteq A'' \supseteq A'$

Further foΦ : A″→R is a homomorphism such that Ker foΦ ⊃ A'. But R is a rational extension of A'. Therefore foΦ = 0 and hence f = 0 on B″. That is, B is a rational extension of B'.

Therefore B ⊂ R and B is a rational extension of any of its N-sub modules. As R is uniform, B is also uniform. Therefore B is strongly uniform.

As a corollary to the above, we have

Corollory 3.3

If M satisfies ascending chain condition on N-sub modules and also intersection property, then Ass M ≠ Φ.

Proof: By theorem 1.3.2., M has an N-sub module B which is strongly uniform. Let 0 ≠ x Є B. Then Nx ≅ N/P where P = Ann (x) and Nx is strongly uniform. Hence P is a critical left ideal associated to M. Therefore P Є Ass (m).

Hence Ass (M) ≠ Φ.

Some of the properties are following.

Proposition 3.4

Let M be an N-module satisfying ascending chain condition on N-sub modules.

a) If M is the union of a family of the N-ideals $\{M_i\}$ of M, then

$$\text{Ass } M = \cup_i \text{Ass } M_i.$$

b) If P is a critical left ideal then

$$\text{Ass } (N/P) = \{[P]\}$$

c) If R ⊆ M, then Ass R ⊆ Ass M ⊆ Ass R ∪ Ass (M/R).

d) If M is the direct sum of N-ideals $\{M_i\}$ of M, then

$$\text{Ass } M = \cup_i \text{Ass } M_i.$$

Proof

a) Since $M = \cup_i M_i$, we have $M_i \subseteq M$ every i

=> Ass $M_i \subseteq$ Ass M for every i

=> U_i Ass $M_i \subseteq$ AssM.

Suppose P Є Ass (M), and then P is a critical left ideal belonging to M.

- ⇨ There exists x Є M such that Ann(x) = P where P is critical left ideal. Since $N = \cup M_i$ we have x Є Ni for some i.
- ⇨ Ann(x) = P where P is critical left ideal.
- ⇨ P Є Ass (M_i)
- ⇨ Ass (M) ⊆ Ass (M_i)
- ⇨ Ass M ⊆ U_i Ass (M_i)

Therefore Ass M = U_i Ass M_i.

b) We have to show that Ass (N/P) = {[P]} where P is a critical left ideal of N.

Let P be a critical left ideal of N.

Let Q be any element of Ass (N/P).

- ⇨ Q is a critical left ideal belonging to N/P.
- ⇨ There exists a non-zero element x + P in N/P such that

Ann (x + P) = Q.

Since x + P is non-zero we have x∉ P. Since we are assuming that 1 Є N, we have that any critical left ideal is a proper ideal. Hence 1∉ Q. Now we shall show that $Q.1^{-1} = Px^{-1}$.

We Have $Q.1^{-1}$ = {r Є N : r . 1 Є Q} = Q

Now Px^{-1} = { r Є N : r.x Є P}

= {r Є N : rx + P = P}

= {r Є N : r(x + P) = P}

= Ann (x + P) = Q

Therefore $Q. 1^{-1} = Px^{-1}$.

Hence Q and P are related, i.e. Q Є [P]

Conversely suppose that Q Є [P].

- ⇨ Q and P are related.
- ⇨ There exists x and y in N such that

x∉ P and y∉ Q and $Px^{-1} = Qy^{-1}$.

Since x ⊆ P we have x+ P is non-zero element of N/P. It can be verified that Q = Ann (x + P).

- ⇨ Q is a critical left ideal belonging to N/P
- ⇨ Q Є Ass (N/P)

Thus we have Ass (N/P) = {[P]}

c) If R ⊆ M, then Ass(R) ⊆ Ass (M) is clear.

Let R be an N-ideal of M.

Let P Є Ass M and P = Ann (x). 0 ≠ x Є M.

Case i) If Nx ∩ R = <0>

Ann (x + R) in M/R = {y Є N : y(x + R) = 0 + R}
= {y Є N : yx Є R}
= {y Є N : yx Є R ∩ Nx}
= {y Є N : yx = 0}
= Ann (x) in M.
i.e. P Є Ass (M/R)

Case ii)If Nx ∩ R ≠ <0>, Let R' = Nx ∩ R

Consider R' as N-module. Then Ass (R') ≠ Φ by Corollary 1.3.3.

Let Q Є Ass (R')

⇨ There exists y Є R' such that Ann (y) = Q

y Є Nx => There exists a Є N such that y = ax.

Q = Ann (y)
= {z : zy = 0}
= {z : z(ax) = 0}
= {z : (za)x = 0}
= {z : za Є P}
= Pa^{-1}

So $Pa^{-1} = Q1^{-1}$, then P is related to Q.

Hence P Є Ass(R)

4. Primary N-Ideals

Definition 4.1

An N-module M is said to be co primary if Ass (M) consists of a single element.

Definition 4.2

An N-ideal R of M is said to be primary if quotient module M/R is co primary.

Theorem 4.3

Suppose M is an N-module which satisfies A.C.C. on N-sub modules and uniform, then M is co primary.

Proof: Let P Є Ass M, P = Ann (x) and
Nx ≡ N/P

If Q Є Ass M, then Q = Ann (y), Ny ≡ N/Q

Since M is uniform Nx ∩ Ny ≠ 0

Let C = Nx ∩ Ny. Then C is isomorphic to a non-zero N-submodule of N/P and also a non-zero N-sub module of N/Q. Therefore P is related to Q.

Q ~ P Є Ass M.

Therefore Ass (m) = {[P]}.

Definition 4.4

Let R be an N-ideal of M. R is said to be "irreducible" if for any two N-ideals R_1 and R_2 of M, $R\subset R_1$, $R\subset R_2 \Rightarrow R\subset R_1 \cap R_2$.

Definition 4.5

R is said to be "strongly irreducible" if for any two N-sub modules R_1 and R_2, $R\subset R_1$, $R\subset R_2 \Rightarrow R\subset R_1 \cap R_2$.

If M satisfies intersection property, clearly an N-ideal

R is irreducible if and only if it is strongly irreducible.

Hence if R is an irreducible N-ideal then M/R is uniform.

Theorem 1.4.6

If M is an N-module which satisfies A.C.C. on N-sub modules then every N-ideal of M is a finite intersection of irreducible N-ideals.

This can be proved by using A.C.C. on N-sub modules.

Definition 4.7

A primary decomposition of an N-ideal R of M is a representation of R as a finite intersection of primary N-ideals.

Theorem 4.8

If M is an N-module satisfying A.C.C. on N-sub modules and intersection property, then every N-ideal R of M has a primary decomposition.

Proof: By theorem 1.4.6., an N-ideal R of M can be written as $R = R_1 \cap R_2 \cap \ldots \cap R_k$ where each R_i is irreducible N-ideal. By the intersection property each R_i is strongly irreducible. Then for each i, M/R_i is uniform and satisfies A.C.C. on N-sub modules and it is co primary by theorem 1.4.3. Thus R_i is primary.

5. Z-S-Coprimary N-Ideals

It can be shown that every primary decomposition has reduced primary decomposition and uniqueness by the familiar methods.

Lemma 5.1

Let N be a near-ring with 1 satisfying the conditions

a)Possesses A.C.C. on ideals.

b)Every N-sub modules of N is an N-ideal on N Then every critical

Left ideal of N is a prime ideal.

Proof: By condition (b), P is an ideal of N and by definition of critical left ideal; N/P is strongly uniform. If P is not prime, there exist ideals A and B such that $P\subset A$, $P\subset B$ and $BA\subset P$. Since $P\neq A$, there exists a Є a and $a\notin P$.

Define a mapping f:N/P→N/P as follows f(x + P) = xa + P

It is well-defined because

x + P = y + P

=> x –y Є P

=> (x-y)aЄ P

=> xa-yaЄ P

=> xa + P = ya + P

Clearly this is a module homomorphism and Ker f ⊃ B/P and f ≠ 0. For if f = 0, then xa Є P for every x Є N i.e. Na ⊂ P which is not the case since a Є Na and a ∉ P.

This is a contradiction to the hypothesis that N/P is a rational extension of each of its non-zero N-submodules.

Hence P is prime ideal.

Definition 5.2

An ideal Q of N is said to be Z-S-co primary, if Px ⊂ Q=>P ⊂ r(Q) or x Є Q where r(Q) is the intersection of all prime ideals of N containing Q.

Theorem 5.3

If N satisfies conditions above, then an ideal Q which is Z-S-Co primary is a co primary N-sub module.

Proof: P Є Ass (N/Q) and P = Ann (x + Q)

That is Px ⊂ Q, x∉ Q. Therefore P ⊂ r(Q)

But r(Q) ⊂ P (since P is prime)

Therefore P = r(Q) which implies

Ass (N/Q) = {[P]}.

CONCLUSION

Our emphasis has been the study of primary decomposition of left n-modules over Right near-rings that are given by small number of generators but are potentially very big. Various efficient algorithms for problems in this area have been developed. Based on these, some interesting N/P is a rational extension of each of its non- zero N sub modules which are strongly uniform. The results in this article should be considered as a solid basis for further investigations.

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